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Original Article

# Investigate and Designing Translation surface with a 2-Bishop frame in the Galilean space.

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**Abstract:**

In this study, we will study the translation surface in Galilean space with a 2-Bishop frame. For this surface, we have determined the first and second fundamental forms. we concentrated on the distinctive characteristics of the translational surface (Gaussian curvatures $\mathbb{K}(\mathcal{P},\mathfrak{D})$ and mean curvatures $\mathbb{H}(\mathcal{P},\mathfrak{D})$). The Weingarten property requirement for the Gaussian and second Gaussian curvatures was then satisfied ($(\mathbb{K}_{II},\mathbb{H})$-Weingarten translation surface and $(\mathbb{K}_{II},\mathbb{H})$-Weingarten translation surface). We will study some properties of this surface as minimal Translation surface and developable Translation surface.

**Keywords:** Translation surface, 2-Bishop frame, Galilean space.

## 1. Introduction

Polygons provide a simple definition of translation surfaces, which naturally appear while studying many fundamental dynamical systems. They have moduli spaces known as strata that are connected to the moduli space of Riemann surfaces and may also be characterized as differentials on Riemann surfaces. Each stratum has a $\mathrm{GL}(2,\mathrm{R})$ action, and in order to solve the majority of translation surface issues, one must first be aware of how this action closes the orbit of the translation surface. These orbit closures also have fundamental relevance on their own and are now understood to be algebraic varieties that parameterize translation surfaces with exceptional flat and algebra-geometric qualities. Recent years have seen a sharp increase in the study of orbit closures, with new methods and concepts emerging from several mathematics branches (Aydın et al., 2022).

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At practically every educational level, translation surfaces are an excellent instructional tool. You are limited at the primary or secondary level by the absence of embedding into 3-space, but pool offers several entry points into geometry. In topology, they aid in the explanation of quotient spaces; in differential geometry, they aid in the explanation of atlases and transition maps; or in complex analysis, they aid in the explanation of meromorphic functions. The gates of abstraction break open at the graduate level, allowing you to go in a few easy steps to the enchanting gardens of moduli spaces, Teichmller theory, and fundamental algebraic geometry. The subject's beauty is found in the way that the fundamental (cutting and pasting tiny bits of paper) and the abstract interact (moduli spaces of Riemann surfaces) (Gun Bozok & Oztekin, 2019).

L.R. Bishop introduced the Bishop frame, also known as an alternate or parallel frame of the curves, in 1975 using parallel vector fields. Several research papers on this topic have recently been published in Euclidean space (see (B. Bükcü & Murat Kemal Karacan, 2009; Sepet et al., 2022) ); in Minkowski space,(see (Bükcü & Karacan, 2007; B. Bükcü & M. K. Karacan, 2009; Karacan & Bukcu, 2008; Yılmaz, 2009))

Special Relativity and Galilean Relativity's geometry are connected by a "bridge" in Euclidean geometry. The space of Galilean Relativity can be described as the Galilean space in three-dimensional projective space $P_3(\mathbb{R})$. Although there are parallels between the pseudo-Galilean and Galilean geometries, they are obviously distinct from one another (Aydin et al., 2015).

One Non-Euclidean geometry that is crucial to Special Relativity is Galilean geometry. One can read to learn more about Galilean geometry (Artıkbayev et al., 2013).

Galilean geometry is the geometry that is used to apply special relativity to Euclidean geometry. Long ago, researchers looked at surfaces and curves in Euclidean space. Galilean spaces $G_3$ and $G_4$ have recently seen the introduction of curves and surfaces by mathematicians (Elzawy, 2021).

In this paper we will study the translation surface in Galilean space with a 2-Bishop frame, The translation surface in $G_3$ will be taken into consideration, and the Gauss curvature $\mathbb{K}(\mathcal{P},\mathfrak{D})$and Mean curvature $\mathbb{H}(\mathcal{P},\mathfrak{D})$ of the translation surfac $\mathfrak{R}(\mathcal{P},\mathfrak{D})$will be studied. we invsetgate the $(\mathbb{K}_{II},\mathbb{H})$-Weingarten translation surface and the $(\mathbb{K},\mathbb{H})$-Weingarten translation surface in Galilean $G_3$ .We investigate a specific Weingarten type of Translation surface in Galilean space that satisfies certain interesting and significant equations in terms of the

Gaussian curvature, the mean curvature, and the second Gaussian curvature. We will study some properties of this surface as minimal Translation surface, developable Translation surface, and Weingarten Translation surface.

## 1. Basic concepts:

The references (Aydın et al., 2022; Elzawy, 2021; Gun Bozok & Oztekin, 2019) and (Sepet et al., 2022) the Cayley-Klein model, which is equipped with the metric of signature $(0, 0, +, +)$, also known as a projective metric, defines -the Galilean space of dimension three $(G_3)$ as the space. The triple$(\mu, \eta, \xi)$ , known as the "absolute" of Galilean geometry, where $\mu$ is defined as the ideal plane (also known as the absolute plane), $\eta$ is a line in the absolute plane that is known as the absolute line, and $\xi$ is defined as the elliptic involution point $(0, 0, \delta_2, \delta_3) \rightarrow (0, 0, \delta_3, -\delta_2)$.

The term "Euclidean plane" refers to a plane that includes $\eta$, whereas the term "isotropic plane" refers to a plane that does not contain $\eta$. Therefore, planes with $\delta =$ constant are Euclidean planes, i.e., the plane $\mu$ is a Euclidean plane.

If the component $\varkappa_1$ is not equal to zero, the vector $\varkappa = (\varkappa_1, \varkappa_2, \varkappa_3)$ is referred to as a non-isotropic vector. The vectors with the formula $(1, \varkappa_2, \varkappa_3)$ are all unit non-isotropic vectors. The isotropic vectors with the values $\varkappa = (0, \varkappa_2, \varkappa_3)$.

Galilean space $G_3$ has four different sorts of lines.

1. Proper non-isotropic lines are those that do not cross the absolute line $\eta$.
2. The correct isotropic lines are those that cross the absolute line $\eta$ but do not belong to the ideal plane.
3. All perfect plane lines, with the exception of $\eta$, are known as proper non-isotropic lines.
4. the unwavering line $\eta$.

Consider two vectors in Galilean space $G_3$ with the values of $\vec{y} = (y_1, y_2, y_3)$ and $\vec{\varkappa} = (\varkappa_1, \varkappa_2, \varkappa_3)$. In $G_3$, the Galilean scalar product is

$$\langle \vec{y}, \vec{\varkappa} \rangle = \begin{cases} y_1 \varkappa_1 & if\ \varkappa_1 \neq 0\ or\ y_1 \neq 0 \\ \varkappa_2 y_2 + y_3 \varkappa_3 & if\ \varkappa_1 = 0\ and\ y_1 = 0 \end{cases} \quad (1)$$

The vector's norm, $\vec{y} = (y_1, y_2, y_3)$, may be expressed as

$$\|\vec{y}\| = \sqrt{\langle \vec{y}, \vec{y} \rangle} \quad (2)$$

In Galilean space $G_3$, the vector product of $\vec{y} = (y_1, y_2, y_3)$ and $\vec{\varkappa} = (\varkappa_1, \varkappa_2, \varkappa_3)$ is given by

$$\vec{\mathcal{Y}} \times \vec{\varkappa} = \begin{cases} \begin{pmatrix} 0 & \wp_2 & \wp_3 \\ \mathcal{Y}_1 & \mathcal{Y}_2 & \mathcal{Y}_3 \\ \varkappa_1 & \varkappa_2 & \varkappa_3 \end{pmatrix} & if\ \varkappa_1 \neq 0\ or\ \mathcal{Y}_1 \neq 0 \\ \begin{pmatrix} \wp_1 & \wp_2 & \wp_3 \\ \mathcal{Y}_1 & \mathcal{Y}_2 & \mathcal{Y}_3 \\ \varkappa_1 & \varkappa_2 & \varkappa_3 \end{pmatrix} & if\ \varkappa_1 = 0\ and\ \mathcal{Y}_1 = 0 \end{cases} \tag{3}$$

The Serret frame and type-2-Bishop frame are denoted by the symbols, $\{\wp_1, \wp_2, \wp_3\}$ and $\{\varrho_1, \varrho_2, \varrho_3\}$ respectively, along the unit speed curve $\mathcal{S}(\mathcal{P})$.(see (Özyılmaz, 2011))

The vectors $\wp_1, \wp_2\ and\ \wp_3$ are known as the vectors of tangent, primary normal, and binormal of $\mathcal{S}(\mathcal{P})$, respectively. The following Frenet formula is applicable to their derivatives $\mathcal{S}(\mathcal{P})$

$$\begin{cases} \dfrac{d\wp_1}{d\mathcal{P}} = \kappa(\mathcal{P})\ \wp_2(\mathcal{P}), \\ \dfrac{d\wp_2}{d\mathcal{P}} = -\ \tau\ \wp_3(\mathcal{P}), \\ \dfrac{d\wp_3}{d\mathcal{P}} = -\tau\ \wp_2(\mathcal{P}). \end{cases} \tag{4}$$

where $\kappa$ and $\tau$ represent the curvature and torsion of $\mathcal{S}(\mathcal{P})$, respectively

$$\mathrm{B}(\wp_1, \wp_1) = 1,\ \ B(\wp_2\ , \wp_2) = 1,\ \ B(\wp_3\ , \wp_3) = 1,$$

$$\mathrm{B}(\wp_1,\ \ \wp_2) =\ B(\wp_2, \wp_3) = B(\wp_1, \wp_3) = 0.$$

The formula for the type 2-Bishop frame is as follows:

$$\begin{cases} \varrho_1{}'(\mathcal{P})\ = -\kappa_1(\mathcal{P})\ \varrho_3(\mathcal{P})\ , \\ \varrho_2{}'(\mathcal{P})\ = -\kappa_2(\mathcal{P})\ \varrho_3(\mathcal{P}), \\ \varrho_3{}'(\mathcal{P})\ = \kappa_2(\mathcal{P})\ \varrho_2(\mathcal{P}). \end{cases} \tag{5}$$

where

$$\mathrm{M}(\varrho_1\ , \varrho_1) = 1,\ \ M(\varrho_2\ , \varrho_2) = 1,\ \ M(\varrho_3\ , \varrho_3) = 1,$$

$$\mathrm{M}(\varrho_1\ , \varrho_2) =\ \mathrm{M}(\varrho_2\ , \varrho_3) = \mathrm{M}(\varrho_1\ , \varrho_3) = 0$$

The Bishop curvatures are defined in this case by

$$\kappa_1(\mathcal{P}) = \kappa(\mathcal{P})\ \cos\Gamma\,(\mathcal{P}), \qquad and\ \ \kappa_2(\mathcal{P}) = \kappa(\mathcal{P})\ \sin\Gamma\,(\mathcal{P}) \tag{6}$$

where

$$\Gamma = \tan^{-1}\left(\frac{\kappa_2(\mathcal{P})}{\kappa_1(\mathcal{P})}\right), \qquad \tau = \Gamma', and \;\; \kappa(\mathcal{P}) = \sqrt{\kappa_1^2(\mathcal{P}) + \kappa_2^2(\mathcal{P})}, \qquad (7)$$

This equation explains the relationship and differentiates between Frenet and type-2-Bishop frames (Al-Dayel & Solouma, 2021; Gun Bozok & Oztekin, 2019; Özyılmaz, 2011).

$$\begin{pmatrix} \wp_1(\mathcal{P}) \\ \wp_2(\mathcal{P}) \\ \wp_3(\mathcal{P}) \end{pmatrix} = \begin{pmatrix} sin\,\Gamma\,(\mathcal{P}) & -\,cos\,\Gamma\,(\mathcal{P}) & 0 \\ cos\,\Gamma\,(\mathcal{P}) & sin\,\Gamma\,(\mathcal{P}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varrho_1(\mathcal{P}) \\ \varrho_1(\mathcal{P}) \\ \varrho_1(\mathcal{P}) \end{pmatrix} \qquad (8)$$

In $I^3$, we created a surface $\mathfrak{R}$ by

$$\mathfrak{R}(\mathcal{P},\mathfrak{D}) = \left(\mathfrak{R}_1(\mathcal{P},\mathfrak{D}),\;\; \mathfrak{R}_2(\mathcal{P},\mathfrak{D}),\;\; \mathfrak{R}_3(\mathcal{P},\mathfrak{D})\right), \qquad (9)$$

A surface $\mathcal{A}$ in $G_3$ has the following first fundamental form: see (Soliman et al., 2019)

$$\mathrm{I} = \mathcal{R}_{11}\, d\mathcal{P}^2 + \mathcal{R}_{12}\, d\mathcal{P}\, d\mathfrak{D} + \mathcal{R}_{22}\, d\mathfrak{D}^2 \qquad (10)$$

where

$$\mathcal{R}_{11} = \langle \mathfrak{R}_{\mathcal{P}},\;\; \mathfrak{R}_{\mathcal{P}} \rangle, \qquad \mathcal{R}_{12} = \langle \mathfrak{R}_{\mathcal{P}},\;\; \mathfrak{R}_{\mathfrak{D}} \rangle, \qquad \mathcal{R}_{22} = \langle \mathfrak{R}_{\mathfrak{D}},\;\; \mathfrak{R}_{\mathfrak{D}} \rangle \qquad (11).$$

A completely galilean translation surface's normal vector field $\mathcal{Q}$ is described by the $II$ by

$$II = \mathcal{W}_{11}\, d\mathcal{P}^2 + \mathcal{W}_{12}\, d\mathcal{P}\, d\mathfrak{D} + \mathcal{W}_{22}\, d\mathfrak{D}^2 \qquad (12)$$

where

$$\mathcal{W}_{11} = \langle \mathfrak{R}_{\mathcal{P}\mathcal{P}}, \mathcal{Q} \rangle, \quad \mathcal{W}_{12} = \langle \mathfrak{R}_{\mathcal{P}\mathfrak{D}}, \mathcal{Q} \rangle, \quad \mathcal{W}_{22} = \langle \mathfrak{R}_{\mathfrak{D}\mathfrak{D}}, \mathcal{Q} \rangle \quad (13).$$

Gaussian curvature K, mean curvature H, and second Gaussian curvature $\mathbb{K}_{II}$ can be defined as follows,

$$\mathbb{H}(\mathcal{P},\mathfrak{D}) = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{\mathcal{R}_{11}\mathcal{W}_{22} - 2\mathcal{R}_{12}\mathcal{W}_{12} + \mathcal{R}_{22}\mathcal{W}_{11}}{2(\mathcal{R}_{11}\mathcal{R}_{22} - \mathcal{R}_{12}^2)} \qquad (15)$$

$$\mathbb{K}_{II}(\mathcal{P},\mathfrak{D}) = \frac{1}{(\mathcal{W}_{11}\mathcal{W}_{22} - \mathcal{W}_{12}^2)^2}\left\{\begin{vmatrix} -\frac{1}{2}\mathcal{W}_{11_{\mathfrak{D}\mathfrak{D}}} + \mathcal{W}_{12_{\mathcal{P}\mathfrak{D}}} - \frac{1}{2}\mathcal{W}_{22_{\mathcal{P}\mathcal{P}}} & \frac{1}{2}\mathcal{W}_{11_{\mathcal{P}}} & \mathcal{W}_{12_{\mathcal{P}}} - \frac{1}{2}\mathcal{W}_{11_{\mathfrak{D}}} \\ \mathcal{W}_{12_{\mathfrak{D}}} - \frac{1}{2}\mathcal{W}_{22_{\mathcal{P}}} & \mathcal{W}_{11} & \mathcal{W}_{12} \\ \frac{1}{2}\mathcal{W}_{22_{\mathfrak{D}}} & \mathcal{W}_{12} & \mathcal{W}_{22} \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}\mathcal{W}_{22_{\mathcal{P}}} & \frac{1}{2}\mathcal{W}_{22_{\mathfrak{D}}} \\ \frac{1}{2}\mathcal{W}_{22_{\mathcal{P}}} & \mathcal{W}_{11} & \mathcal{W}_{12} \\ \frac{1}{2}\mathcal{W}_{22_{\mathfrak{D}}} & \mathcal{W}_{12} & \mathcal{W}_{22} \end{vmatrix}\right\}. \quad (16)$$

**The Galilean space contains a translation surface with a 2-Bishop frame.**

Let $\mathcal{A}: \mathfrak{R}(\mathcal{P},\mathfrak{D}) = \mathcal{S}(\mathcal{P}) + \mathcal{Q}(\mathfrak{D})$ be a translation surface ,where $\mathcal{S}(\mathcal{P}), \mathcal{Q}(\mathfrak{D})$are two curves in the surface . in this paper we study the Translation surface generated by the center curve $\mathcal{Q}(\mathfrak{D})$ and the spin curve $\mathcal{S}(\mathcal{P})$ . In this case, the surface becomes as follows

$$\mathfrak{R}(\mathcal{P},\mathfrak{D}) = \mathcal{S}(\mathcal{P}) + \cos(\mathfrak{D})\,\wp_2 + \sin(\mathfrak{D})\,\wp_3 \qquad (17)$$

Where

$$\mathcal{S}'(\mathcal{P}) = \mathcal{P}\,T(\mathcal{P}) \qquad (18)$$

**The properties of the translation surface in Galilean space:**

The 1st derivative of the translation surface are

$$\left.\begin{aligned} \mathfrak{R}_{\mathcal{P}} &= \{\mathcal{P}, \sin(\mathfrak{D})\,k_2(\mathcal{P}), -\cos(\mathfrak{D})k_2(\mathcal{P})\} \\ \mathfrak{R}_{\mathfrak{D}} &= \{0, -\sin(\mathfrak{D}), -\cos(\mathfrak{D})\} \end{aligned}\right\} \qquad (19)$$

The first fundamental form of the translation surface are

$$\mathcal{R}_{11} = \mathcal{P}^2 \quad , \quad \mathcal{R}_{12} = -k_2(\mathcal{P}) \quad , \quad \mathcal{R}_{22} = -k_2(\mathcal{P}) \qquad (20)$$

The 2nd derivative of the translation surface are

$$\left.\begin{aligned} \mathfrak{R}_{\mathcal{P}\mathcal{P}} &= \{1, -\cos(\mathfrak{D})k_2(\mathcal{P})^2 + \sin(\mathfrak{D})\,k_2{}'(\mathcal{P}), -\mathcal{P}k_1(\mathcal{P}) - \sin(\mathfrak{D})\,k_2(\mathcal{P})^2 - \cos(\mathfrak{D})k_2{}'(\mathcal{P})\} \\ \mathfrak{R}_{\mathfrak{D}\mathfrak{D}} &= \{0, -\cos(\mathfrak{D}), -\sin(\mathfrak{D})\} \\ \mathfrak{R}_{\mathcal{P}\mathfrak{D}} &= \{0, \cos(\mathfrak{D})k_2(\mathcal{P}), \sin(\mathfrak{D})\,k_2(\mathcal{P})\} \end{aligned}\right\} \qquad (21)$$

The Galilean unit normal vector field is

$$\mathcal{Q} = \frac{\mathfrak{R}_{\mathcal{P}} \times \mathfrak{R}_{\mathfrak{D}}}{|\mathfrak{R}_{\mathcal{P}} \times \mathfrak{R}_{\mathfrak{D}}|} = \{0, -\cos(\mathfrak{D}), -\sin(\mathfrak{D})\} \qquad (22)$$

The second fundamental form of the translation surface are

$$\mathcal{W}_{11} = \mathcal{P}\sin(\mathfrak{D})\,k_1(\mathcal{P}) + k_2(\mathcal{P})^2 \quad , \quad \mathcal{W}_{12} = -k_2(\mathcal{P}) \quad , \quad \mathcal{W}_{22} = 1 \tag{23}$$

The Galilean mean curvature , The Galilean gaussian curvature and the second Galilean gaussian curvature are [see Figure .1 and Figure .2 ]

$$\mathbb{K} = -\frac{\mathcal{P}\sin(\mathfrak{D})\,k_1(\mathcal{P})}{k_2(\mathcal{P})\big(\mathcal{P}^2 + k_2(\mathcal{P})\big)} \tag{24}$$

$$\mathbb{H} = \frac{-\mathcal{P}^2 + \mathcal{P}\sin(\mathfrak{D})\,k_1(\mathcal{P})\,k_2(\mathcal{P}) + k_2[\mathcal{P}]^2(2 + k_2(\mathcal{P}))}{2k_2(\mathcal{P})(\mathcal{P}^2 + k_2(\mathcal{P}))} \tag{25}$$

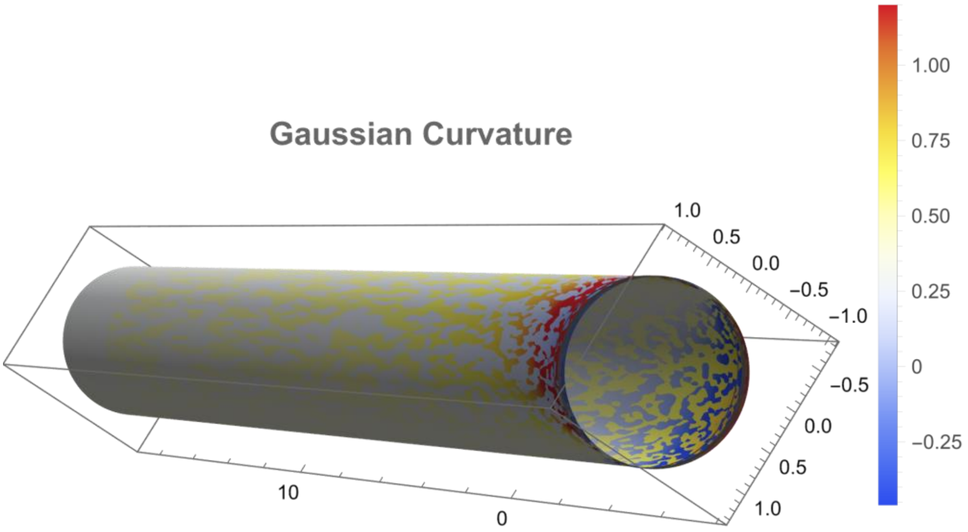


**Figure .1** The maximum value of Galilean gaussian curvature is in the red color and the minimum value in blue color [see Eq. (25)].

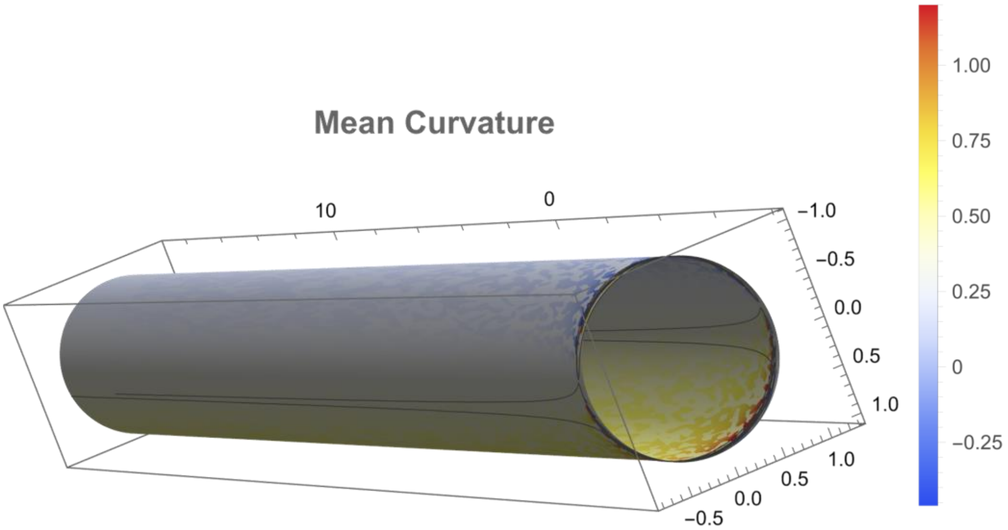


**Figure .2** The maximum value of Galilean mean curvature is in the red color and the minimum value in blue color [see Eq. (24)].

$$\mathbb{K}_{II} = -\frac{1}{8}\left(-3 + \cos(2\mathfrak{D})\right)\csc(\mathfrak{D})^2 \qquad (26)$$

## 2. Main Result

In this section, we will study some properties of the translation surface, such as: developable Galilean surface , minimal Galilean surface , Weingarten Galilean surface

**Theorem 1**: A translation surface with 2-bishop frame in Galilean space is minimal if

$$k_1[\mathcal{P}] = \frac{\csc(\mathfrak{D})\left(\mathcal{P}^2 - 2k_2(\mathcal{P})^2 - k_2(\mathcal{P})^3\right)}{\mathcal{P}k_2(\mathcal{P})}$$

Proof : A translation surface $\mathcal{A}$ satisfies the condition $\mathbb{H} = 0$ ,we substations in Eq.(24)

$$\frac{-\mathcal{P}^2 + \mathcal{P}\sin(\mathfrak{D})\,k_1(\mathcal{P})k_2(\mathcal{P}) + k_2[\mathcal{P}]^2\left(2 + k_2(\mathcal{P})\right)}{2k_2(\mathcal{P})(\mathcal{P}^2 + k_2(\mathcal{P}))} = 0$$

Then we have

$$-\mathcal{P}^2 + \mathcal{P}\sin(\mathfrak{D})\,k_1(\mathcal{P})k_2(\mathcal{P}) + k_2(\mathcal{P})^2\left(2 + k_2(\mathcal{P})\right) = 0$$

$$\therefore k_1(\mathcal{P}) = \frac{\csc(\mathfrak{D})\left(\mathcal{P}^2 - 2k_2(\mathcal{P})^2 - k_2(\mathcal{P})^3\right)}{\mathcal{P}k_2(\mathcal{P})}$$

A translation surface $\mathcal{A}$ is minimal

**Theorem 2**: A translation surface with 2-bishop frame in Galilean space is developable if

$$k_1(\mathcal{P}) = 0$$

Proof : A translation surface $\mathcal{A}$ satisfies the condition $\mathbb{K} = 0$ ,we substations in Eq.(25)

$$-\frac{\mathcal{P}\sin(\mathfrak{D})\,k_1(\mathcal{P})}{k_2(\mathcal{P})\left(\mathcal{P}^2 + k_2(\mathcal{P})\right)} = 0$$

we get

$$\mathcal{P}\sin(\mathfrak{D})\,k_1(\mathcal{P}) = 0$$

Then we have

$k_1(\mathcal{P}) = 0$ or $\mathcal{P} = 0$

A translation surface $\mathcal{A}$ is developable.

**Theorem 3**: A translation surface with 2-bishop frame in Galilean space is $(\mathbb{K}, \mathbb{H})$-Weingarten translation surface if $\mathcal{P}$

1- $k_1(\mathcal{P}) = 0$

Or

2- $k_1(\mathcal{P}) = \frac{1}{\left(\mathcal{P}k_2(\mathcal{P})\left(\mathcal{P}^2 + k_2(\mathcal{P})\right)k_2{}'(\mathcal{P})\right)}$

$$(\csc(\mathfrak{D})\,(2\mathcal{P}k_2(\mathcal{P})^2+4\mathcal{P}k_2(\mathcal{P})^3+2\mathcal{P}k_2(\mathcal{P})^4-\mathcal{P}^4k_2{}'(\mathcal{P})-2\mathcal{P}^2k_2(\mathcal{P})k_2{}'(\mathcal{P})-2\mathcal{P}^2k_2(\mathcal{P})^2k_2{}'(\mathcal{P}) - 2\mathcal{P}^2k_2(\mathcal{P})^3k_2{}'(\mathcal{P})-k_2(\mathcal{P})^4k_2{}'(\mathcal{P})))$$

Proof: we drive the Eq. (24) ,(25)

$$\left.\begin{array}{c}
\mathbb{H}_{\mathcal{P}}=\frac{1}{(2k_2(\mathcal{P})^2(\mathcal{P}^2+k_2(\mathcal{P}))^2}\left(\begin{array}{c}\mathcal{P}^4k_2{}'(\mathcal{P})+2\mathcal{P}^2k_2(\mathcal{P})k_2{}'(\mathcal{P})+k_2(\mathcal{P})^4\\(-2\mathcal{P}+k_2{}'(\mathcal{P}))+\mathcal{P}k_2(\mathcal{P})^2(-2+\mathcal{P}^2\sin(\mathfrak{D})\,k_1{}'(\mathcal{P})+\\2\mathcal{P}k_2{}'(\mathcal{P})-\sin(\mathfrak{D})\,k_1(\mathcal{P})(\mathcal{P}+k_2{}'(\mathcal{P})))\\+k_2(\mathcal{P})^3\left(\sin(\mathfrak{D})\,k_1(\mathcal{P})+\mathcal{P}\left(-4+\sin(\mathfrak{D})\,k_1{}'(\mathcal{P})+2\mathcal{P}k_2{}'(\mathcal{P})\right)\right)\end{array}\right)\\
\mathbb{H}_{\mathfrak{D}}=\frac{\mathcal{P}\cos(\mathfrak{D})k_1(\mathcal{P})}{2(\mathcal{P}^2+k_2(\mathcal{P}))}\\
\mathbb{K}_{\mathcal{P}}=\frac{1}{(k_2(\mathcal{P})^2(\mathcal{P}^2+k_2(\mathcal{P}))^2}\left(\begin{array}{c}\sin(\mathfrak{D})\,(-\mathcal{P}k_2(\mathcal{P})(\mathcal{P}^2+k_2(\mathcal{P}))k_1{}'(\mathcal{P})\\+k_1(\mathcal{P})(-k_2(\mathcal{P})^2+\mathcal{P}^3k_2{}'(\mathcal{P})+\mathcal{P}k_2(\mathcal{P})(\mathcal{P}+2k_2{}'(\mathcal{P}))))\end{array}\right)\\
\mathbb{K}_{\mathfrak{D}}=-\frac{\mathcal{P}\cos(\mathfrak{D})k_1(\mathcal{P})}{k_2(\mathcal{P})(\mathcal{P}^2+k_2(\mathcal{P}))}
\end{array}\right\} \quad (27)$$

A $(\mathbb{K},\mathbb{H})$-Weingarten translation surface $\mathfrak{R}(\mathcal{P},\mathfrak{D})$ satisfies Jacobi equation

$$\mathbb{H}_{\mathcal{P}}\mathbb{K}_v-\mathbb{H}_v\mathbb{K}_{\mathcal{P}}=0$$

From (27) we have

$$\left(\frac{1}{2k_2(\mathcal{P})^2(\mathcal{P}^2+k_2(\mathcal{P}))^3}(\mathcal{P}\cos(\mathfrak{D})\sin(\mathfrak{D})\,k_1(\mathcal{P})(-\mathcal{P}k_2(\mathcal{P})(\mathcal{P}^2+k_2(\mathcal{P}))k_1{}'(\mathcal{P}) + k_1(\mathcal{P})(-k_2(\mathcal{P})^2+\mathcal{P}^3k_2{}'(\mathcal{P})+\mathcal{P}k_2(\mathcal{P})(\mathcal{P}+2k_2{}'(\mathcal{P}))))) + \frac{1}{2k_2(\mathcal{P})^3(\mathcal{P}^2+k_2(\mathcal{P}))^3}(\mathcal{P}\cos(\mathfrak{D})k_1(\mathcal{P})(\mathcal{P}^4k_2{}'(\mathcal{P})+2\mathcal{P}^2k_2(\mathcal{P})k_2{}'(\mathcal{P}) + k_2(\mathcal{P})^4(-2\mathcal{P}+k_2{}'(\mathcal{P}))+\mathcal{P}k_2(\mathcal{P})^2(-2+\mathcal{P}^2\sin(\mathfrak{D})\,k_1{}'(\mathcal{P})+2\mathcal{P}k_2{}'(\mathcal{P}) - \sin(\mathfrak{D})\,k_1(\mathcal{P})(\mathcal{P}+k_2{}'(\mathcal{P})))+k_2(\mathcal{P})^3(\sin(\mathfrak{D})\,k_1(\mathcal{P})+\mathcal{P}(-4 + \sin(\mathfrak{D})\,k_1{}'(\mathcal{P})+2\mathcal{P}k_2{}'(\mathcal{P})))))\right)=0 \quad (28)$$

By using the Mathematica program solve Eq.(28) Then we get

$$k_1(\mathcal{P})=0$$

Or

$$k_1(\mathcal{P})=\frac{1}{(\mathcal{P}k_2(\mathcal{P})(\mathcal{P}^2+k_2(\mathcal{P}))k_2{}'(\mathcal{P}))}(\csc(\mathfrak{D})(2\mathcal{P}k_2(\mathcal{P})^2+4\mathcal{P}k_2(\mathcal{P})^3+2\mathcal{P}k_2(\mathcal{P})^4-\mathcal{P}^4k_2{}'(\mathcal{P}) - 2\mathcal{P}^2k_2(\mathcal{P})k_2{}'(\mathcal{P})-2\mathcal{P}^2k_2(\mathcal{P})^2k_2{}'(\mathcal{P})-2\mathcal{P}^2k_2(\mathcal{P})^3k_2{}'(\mathcal{P})-k_2(\mathcal{P})^4k_2{}'(\mathcal{P})))$$

A surface $\mathcal{A}$ is $(\mathbb{K},\mathbb{H})$-Weingarten translation surface.

**Theorem 4.** A translation surface with 2-bishop frame in Galilean space is $(\mathbb{K}_{II},\mathbb{H})$-Weingarten translation surface if

$$k_2(\mathcal{P})=0$$

Proof : A $(\mathbb{K}_{II},\mathbb{H})$-Weingarten translation surface $\mathfrak{R}(\mathcal{P},\mathfrak{D})$ satisfies Jacobi equation

$$\mathbb{H}_{\mathcal{P}}(\mathbb{K}_{II})_{\mathfrak{D}} - \mathbb{H}_{\mathfrak{D}}(\mathbb{K}_{II})_{\mathcal{P}} = 0$$

Then we have

$$\left(\frac{1}{(4k_2(\mathcal{P})^2(\mathcal{P}^2+k_2(\mathcal{P}))^2}(\cot(\mathfrak{D})\ \csc(\mathfrak{D})^2\ (\mathcal{P}^4 k_2{}'(\mathcal{P}) + 2\mathcal{P}^2 k_2(\mathcal{P})k_2{}'(\mathcal{P}) + k_2(\mathcal{P})^4(-2\mathcal{P} + k_2{}'(\mathcal{P})) + \mathcal{P}k_2(\mathcal{P})^2(-2 + \mathcal{P}^2 \sin(\mathfrak{D})\, k_1{}'(\mathcal{P}) + 2\mathcal{P}k_2{}'(\mathcal{P}) - \sin(\mathfrak{D})\, k_1(\mathcal{P})(\mathcal{P} + k_2{}'(\mathcal{P}))) + k_2(\mathcal{P})^3(\sin(\mathfrak{D})\, k_1(\mathcal{P}) + \mathcal{P}(-4 + \sin(\mathfrak{D})\, k_1{}'(\mathcal{P}) + 2\mathcal{P}k_2{}'(\mathcal{P}))))\right) = 0$$

we get

$$k_2(\mathcal{P}) = 0$$

Then The translation surface $\mathcal{A}$ is $(\mathbb{K}_{II}, \mathbb{H})$-Weingarten translation surface.

**Conclusion**

In this paper, we studied the translated surface in the Galilee space with a frame 2 Bishop. We found the first and second fundamentel form and Gaussian curvatures $\mathbb{K}(\mathcal{P},\mathfrak{D})$ ,second Gaussian curvature $\mathbb{K}_{II}(\mathcal{P},\mathfrak{D})$and mean curvatures $\mathbb{H}(\mathcal{P},\mathfrak{D})$ for this surface. We studied some properties such as minimal surface under condtion $k_1[\mathcal{P}] = \frac{\csc(\mathfrak{D})(\mathcal{P}^2 - 2k_2(\mathcal{P})^2 - k_2(\mathcal{P})^3)}{\mathcal{P}k_2(\mathcal{P})}$ , developable surface under condtion $k_1(\mathcal{P}) = 0$ , and Wengerten's formula for this surface such as is $(\mathbb{K}_{II}, \mathbb{H})$-Weingarten translation surface $\mathfrak{R}(\mathcal{P},\mathfrak{D})$ under condition $k_2(\mathcal{P}) = 0$ and $(\mathbb{K}, \mathbb{H})$-Weingarten translation surface $\mathfrak{R}(\mathcal{P},\mathfrak{D})$ under condtion $k_1(\mathcal{P}) = 0$

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